KATO'S INEQUALITY AND LIOUVILLE THEOREMS ON LOCALLY FINITE GRAPHS

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ABSTRACT. In this paper we study the Kato' inequality on locally finite graph. We also study the application of Kato inequality to Ginzburg-Landau equations on such graphs. Interesting properties of Schrodinger equation and a Liouville type theorem are also derived.

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1. Introduction

In recent studies Yau and F.Chung and their friends (see [1], [2] and [3] for more background and references) have studied Ricci curvature and eignevalue estimate on locally finite graphs. The lower bound of Ricci curvature on locally finite graphs can be defined via the method of Bakry-Emery. Then following the method of Li-Yau, one can do the gradient estimate for eigen-functions of the Laplacian operators on locally finite graphs. In particular, one can derive the lower bound of the eigenvalues on a connected graph with finite diameter. On a connected graph with finite diameter, one can see that the Liouville theorem is always true for harmonic function. In fact, the harmonic functions are always bounded in the connected finite graphs and their maximum values are obtained somewhere. Using the mean value property, one obtain the Liouville theorem. It is nature to ask if such a Liouville type theorem is true on nonlinear elliptic problems on locally finite graphs. With this question in mind, we intend to study the Kato's inequalities in this paper. As an application we get a Liouville theorem for nonlinear elliptic equations on the locally finite graphs. Our results are stated in lemmas 2.1 and 2.3 below.

We mention the other motivation of this paper. The Ginzbourg-Landau equation is a basic model for the mathematical theory of superconductivity, which examines the macroscopic properties of a superconductor with the aid of general thermodynamic arguments [4]. This equation is derived from the free energy of the form

$$\frac{1}{2}|\nabla u|^2 + \frac{1}{4}(1-|u|^2)^2$$

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of the complex order parameter u. We shall confine the complex variable u defined on locally finite graphs X and study the property of the solutions of the Ginzburg-Landau equation

$$-\Delta u + u(|u|^2 - 1) = 0, \quad in \ \ X.$$

With the help of Kato's inequality we show the uniform bound of the solutions u such that $|u| \leq 1$ on X. Related works on the whole Euclidean space can be found in [5] and [6].

We also study the interesting properties of Schrodinger equations on locally finite graphs.

The plan of the paper is below. Notations are introduced in section 2 and all of our results are stated and proved in section 2.

2. Set up and proofs of main results

Let $(X, \mathcal{E})$ be a graph with countable vertice set X and edge set $\mathcal{E}$. We assume that the graph is *simple*, i.e., no loop and no multi-edges. We also assume that the graph is connected. Let $\mu_{xy} = \mu_{yx} > 0$ is a symmetric weight on $\mathcal{E}$. We call $d_x = \sum_{(x,y)\in\mathcal{E}} \mu_{xy}$ (we also assume $d_x < \infty$ for all $x \in X$) the *degree of* $x \in X$.

Denote by

$$\ell(X) = \{u : \ u : \ X \longrightarrow \mathbb{R}\},$$

the set of all real functions (or complex-valued functions with $\mathbb{R}$ replaced by $\mathbb{C}$ on X. We often denote by u^2 as $|u|^2$.

We define the *Laplacian operator* $\Delta : \ \ell(X) \longrightarrow \ell(X)$:

$$(\Delta u)(x) = \sum_{(x,y)\in\mathcal{E}} \frac{\mu_{xy}}{d_x}\big(u(y) - u(x)\big).$$

We also define

$$|\nabla u|^2(x) = \sum_{(x,y)\in\mathcal{E}} \frac{\mu_{xy}}{d_x}\big(u(y) - u(x)\big)^2.$$

Then we have the following elementary fact.

Lemma 2.1. *(Kato's inequality) For a graph X, we have*

$$|\nabla u|^2 \geq \big|\nabla |u|\big|^2.$$

Proof. For any $x \in X$, we have

$$\begin{aligned} |\nabla u|^2(x) &= \sum_{(x,y)\in\mathcal{E}} \frac{\mu_{xy}}{d_x}\big(u(y) - u(x)\big)^2 \\ &\geq \sum_{(x,y)\in\mathcal{E}} \frac{\mu_{xy}}{d_x}\big(|u(y)| - |u(x)|\big)^2 = |\nabla |u||^2(x). \end{aligned}$$

This completes the proof. $\square$

Generally speaking, given Δu, one may not have the well-defined Δu^2 on fractals. However, this is not the case on graphs.

Lemma 2.2. $\Delta u^2 = 2u\Delta u + |\nabla u|^2$.

Proof. For any $x \in X$,

$$\begin{aligned}(\Delta u^2)(x) &= \sum_{(x,y)\in\mathcal{E}} \frac{\mu_{xy}}{d_x}\big(u^2(y) - u^2(x)\big)\\ &= \sum_{(x,y)\in\mathcal{E}} \frac{\mu_{xy}}{d_x}\big(2u(x)(u(y)-u(x)) + (u(y)-u(x))^2\big)\\ &= 2u(x)\Delta u(x) + |\nabla u|^2(x).\end{aligned}$$

□

With the help of above fact, we have

Lemma 2.3. *(Kato's inequality)*

$$\Delta|u| \quad \geq \operatorname{sign}(u)\Delta u, \tag{2.1}$$

$$\Delta u_+ \quad \geq \operatorname{sign}_+(u)\Delta u. \tag{2.2}$$

Proof. By Lemma 2.2, we have

$$\Delta u^2 = 2u\Delta u + |\nabla u|^2$$

and

$$\Delta u^2 = \Delta |u|^2 = 2|u|\Delta|u| + \big|\nabla|u|\big|^2$$

Hence

$$2|u|\Delta|u| = 2u\Delta u + |\nabla u|^2 - \big|\nabla|u|\big|^2.$$

By Lemma 2.1, we have

$$|u|\Delta|u| \geq u\Delta u$$

It follows that

$$\Delta|u| \geq \frac{u}{|u|}\Delta u = \operatorname{sign}(u)\Delta u,$$

providing $u(x) \neq 0$. If $u(x) = 0$, then

$$\Delta u(x) = \sum_{(x,y)\in\mathcal{E}} \frac{\mu_{xy}}{d_x} u(y) \leq \sum_{(x,y)\in\mathcal{E}} \frac{\mu_{xy}}{d_x}|u(y)| = \Delta|u|(x).$$

we see that (2.1) still hold.

To prove (2.2), we note that $u_+ = \frac{1}{2}(|u| + u)$, hence

$$\begin{aligned}\Delta u_+ &= \frac{1}{2}(\Delta|u| + \Delta u)\\ &\geq \frac{1}{2}\big(\mathrm{sign}(u)\Delta u + \Delta u\big)\\ &= \frac{1}{2}\big(\mathrm{sign}(u) + 1\big)\Delta u\\ &= \mathrm{sign}_+(u)\Delta u.\end{aligned}$$

This completes the proof. □

We now use Kato's inequality to study properties of solutions to the Ginzburg-Landau equation on graphs.

Theorem 2.4. *Assume that u is a solution of the following Ginzburg-Landau equation*

$$\Delta u + u(1 - u^2) = 0, \quad in \quad X.$$

Then $|u| \leq 1$.

Proof. Let $w = u^2 - 1$, then

$$\begin{aligned}\Delta w &= 2u\Delta u + |\nabla u|^2\\ &= 2u \cdot u(u^2 - 1) + |\nabla u|^2\\ &= 2(w + 1)w + |\nabla u|^2.\end{aligned}$$

Hence

$$\begin{aligned}\Delta w_+ &\geq \mathrm{sign}_+(w)\Delta w\\ &\geq \mathrm{sign}_+(w)\big(2(w + 1)w + |\nabla u|^2\big)\\ &\geq 2w_+^2 + 2w_+.\end{aligned}$$

Assume that $\varphi > 0$ such that $-\Delta\varphi = \lambda\varphi$ for some $\lambda > 0$, then

$$0 \leq \int 2(w_+^2 + 2w_+)\varphi \leq \int \varphi\Delta w_+ = \int (\Delta\varphi)w_+ = -\int \lambda\varphi w_+ \leq 0.$$

It follows that $w_+ = 0$, i.e., $w \leq 0$. Hence $u^2 \leq 1$. □

Proposition 2.5. *Assume $Q \geq 0 \in \ell(X)$ and let u be a solution such that*

$$-\Delta u + Qu = 0. \tag{2.3}$$

Then u_+ is a sub-solution of (2.3).

Proof. By the Kato's inequality, we have

$$\Delta u_+ \geq \mathrm{sign}_+(u)\Delta u = \mathrm{sign}_+(u)Qu = Qu_+$$

i.e., $-\Delta u_+ + Qu_+ \leq 0$. That is to say, u_+ is a sub-solution to (2.3). □

with this understanding, we can do the gradient estimate for solutions to the (stationary) Schrodinger equation and our result extends slightly the gradient estimate in [3].

Theorem 2.6. *Assume that* $u,\ Q \in \ell(X)$, $u \geq 0$, $Q \geq 0$, *such that* $-\Delta u + Qu = 0$. *Then*

$$|\nabla u|^2(x) \leq \big(d(1+Q(x))^2 - 2Q(x) - 1\big)\, u^2(x) \leq dQ^2(x)u^2(x), \quad \forall x \in X,$$

where the constant $d = \sup_{x\in X}\sup_{(x,y)\in\mathcal{E}} \frac{d_x}{\mu_{xy}}$.

Proof. Observe that

$$\Delta u(x) = \sum_{(x,y)\in\mathcal{E}} \frac{\mu_{xy}}{d_x}\big(u(y)-u(x)\big) = \sum_{(x,y)\in\mathcal{E}} \frac{\mu_{xy}}{d_x}u(y) - u(x).$$

Hence

$$\sum_{(x,y)\in\mathcal{E}} \frac{\mu_{xy}}{d_x}u(y) = \Delta u(x) + u(x).$$

By definition,

$$\begin{aligned}
|\nabla u|^2(x) &= \sum_{(x,y)\in\mathcal{E}} \frac{\mu_{xy}}{d_x}\big(u(y)-u(x)\big)^2 \\
&= \sum_{(x,y)\in\mathcal{E}} \frac{\mu_{xy}}{d_x}\left(-2u(x)\big(u(y)-u(x)\big) - u^2(x) + u^2(y)\right) \\
&= -2u(x)\sum_{(x,y)\in\mathcal{E}} \frac{\mu_{xy}}{d_x}\big(u(y)-u(x)\big) - u^2(x) + \sum_{(x,y)\in\mathcal{E}} \frac{d_x}{\mu_{xy}}\left(\frac{\mu_{xy}}{d_x}u(y)\right)^2 \\
&\leq -2u(x)\Delta u(x) - u^2(x) + d\left(\sum_{(x,y)\in\mathcal{E}} \frac{\mu_{xy}}{d_x}u(y)\right)^2 \\
&= -(2Q(x)+1)u^2(x) + d\big(\Delta u(x) + u(x)\big)^2 \\
&= -(2Q(x)+1)u^2(x) + d\big(1+Q(x)\big)^2 u^2(x) \\
&= \big(d(1+Q(x))^2 - 2Q(x) - 1\big)\, u^2(x) \\
&\leq dQ^2(x)u^2(x).
\end{aligned}$$

In the first inequality, we have uses $\frac{\mu_{xy}}{d_x}u(y) \geq 0$ for all $y \in X$ such that $(x,y)\in\mathcal{E}$. $\square$

We now derive the Liouville theorem along the line of the Keller-Osserman theory.

Theorem 2.7. *Assume that* $u \in \ell(X)$ *and* $0 \leq u \leq A$ *(where* A *is a positive constant). If* $\Delta u \geq u^p$ *for some* $p \in \mathbb{R}_+$, *then* $u = 0$.

Proof. Suppose otherwise, then there exists $x_0 \in X$ such that $0 < u(x_0) := \rho$. We let $w = \frac{u}{\rho}$. Then $0 \leq w \leq \frac{A}{\rho}$, $w(x_0) = 1$ and

$$\Delta w = \frac{1}{\rho}\Delta u \geq \frac{1}{\rho}u^p = \rho^{p-1}w^p.$$

It follows that, for any $x \in X$,

$$\sum_{(x,y)\in\mathcal{E}} \frac{\mu_{xy}}{d_x}\big(w(y)-w(x)\big) \geq \rho^{p-1} w^p(x).$$

i.e.,

$$\sum_{(x,y)\in\mathcal{E}} \frac{\mu_{xy}}{d_x} w(y) \geq w(x) + \rho^{p-1} w^p(x).$$

Note that the left hand of the above is the (weighted) average of $w(y)$'s. Hence there exists y with $(x,y) \in \mathcal{E}$ such taht

$$w(y) \geq w(x) + \rho^{p-1} w^p(x).$$

Using this and by induction, we get a sequence $\{x_n\}_{n=0}^{\infty} \subset X$ with $(x_i, x_{i+1}) \in \mathcal{E},\ 1 = 0, 1, \cdots$ such that

$$w(x_{n+1}) \geq w(x_n) + \rho^{p-1} w^p(x_n). \tag{2.4}$$

It follows that $\{w(x_n)\}_n$ is a increasing sequence and bounded by constant $\frac{A}{\rho}$. Hence there is a finite limit. Taking the limit at the both side of (2.4), we get $\lim_{n\to\infty} w(x_n) = 0$. This contradicts that the sequence is increasing and $w(x_0) = 1$. This completes the proof. □

We have the following strong maximum principle for the Laplacian equations on the locally finite graph X.

Proposition 2.8. *Assume that $u : X \to \mathbb{R}$ satisfies $\Delta u \geq 0$. If there exists $x_0 \in X$ such that $u(x_0) = \sup_{x\in X} u(x) < \infty$, then u is a constant on X.*

Proof. By the hypothesis on Laplacian, we have

$$u(x_0) \leq \sum_{(x,y)\in\mathcal{E}} \frac{\mu_{xy}}{d_x} u(y) \leq u(x_0).$$

Hence $u(y) = u(x_0)$ for all y such that $(x,y) \in \mathcal{E}$. By induction and the connectivity, we have $u(y) = u(x_0)$ for all $y \in X$. □

Using a similar argument we have

Proposition 2.9. *Assume that $u : \ X \times [0,T] \longrightarrow \mathbb{R}$ such that $u_t = \Delta u$ and $u(x_0,t_0) = \sup\{u(x,t) :\ (x,t) \in X \times [0,T]\},\ t_0 > 0$, then u is constant.*

Proof. At (x_0,t_0), we have $u_t(x_0,t_0) \geq 0$. Similar as above argument, we have $u(y,t_0) = u(x_0,t_0)$ for all $y \in X$ such that $(x_0,y) \in \mathcal{E}$. Also we have $u_t(y,t_0) \geq 0$. Repeating this argument, we see the assertion holds. □

We shall see that the mass and energy conservation laws can also be derived for the Schrodinger equations.

Theorem 2.10. *Assume that the initial data u_0 has finite L^2 norm $||u_0||_{L^2}$ and finite Dirichlet energy $||\nabla u_0||_{L^2}^2$. Then there is a unique solution $u: X\times[0,+\infty):\longrightarrow\mathbb{C}$ to the Schrodinger equation on the locally finite graph X:*

$$iu_t+\Delta u=0;\quad u|_{t=0}=u_0.$$

Then

$$||u(t)||_{L^2}^2=||u_0||_{L^2}^2,\quad ||\nabla u(t)||_{L^2}^2=||\nabla u_0||_{L^2}^2,\quad t\geq 0.$$

Proof. We remark that the existence part of the solution to the Schrodinger equation is by now standard and it can be derived as in the case of heat equation via the fundamental solution. Hence we may omit the detail.

We denote by $(u,v)=u\cdot\bar{v}$ for complex valued functions. Then we have $|u|^2=u\cdot\bar{u}$.

Compute directly and we have

$$\begin{aligned}\frac{d}{dt}||u(t)||_{L^2}^2&=2Re(u,u_t)\\&=-2iIm(u,iu_t)\\&=2iIm(u,\Delta u)\\&=-2iIm(\nabla u,\nabla u)=0.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\frac{d}{dt}||\nabla u||_{L^2}^2&=2Re(\nabla u,\nabla u_t)\\&=2Re(\Delta u,u_t)\\&=2Re(\Delta u,i\Delta u)=0.\end{aligned}$$

Hence the proof is complete.

□

Similar result is true for the Gross-Pitaevskii equation on the finite graph X:

$$iu_t+\Delta u=u(|u|^2-1);\quad u|_{t=0}=u_0$$

with the energy replaced by the free energy

$$\frac{1}{2}|\nabla u|^2+\frac{1}{4}(1-|u|^2)^2$$

and with finite free energy at initial time.

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